Horizons and the Thermal Harmonic Oscillator

M. Cadoni*

Dipartimento di Fisica, Università di Cagliari, and INFN sezione di Cagliari,
Cittadella Universitaria 09042 Monserrato, ITALY

Abstract

We show that two-dimensional anti-de Sitter spacetime (AdS_2) can be put in correspondence, holographically, both with the harmonic oscillator and the free particle. When AdS_2 has a horizon the corresponding mechanical system is a thermal harmonic oscillator at temperature given by the Hawking temperature of the horizon.

*email: mariano.cadoni@ca.infn.it

The quantum description of black holes has represented, since the ground breaking work of S. Hawking, a fundamental challenge for theoretical physics. Classically, a black hole is a gravitating system where information gets lost. At the semiclassical level the black hole emits particles with a thermal spectrum. This feature allows us to compute the entropy of the black hole and to discover that it is proportional to the total area of the black hole horizon. The number of possible quantum states of a black hole should be therefore explained by a field theory residing on the black hole horizon rather then in the volume. What we see here at work is the "Holographic Principle", stating that the total entropy of a system localized in a given region of space is bounded by the an expression proportional to the area of the boundary [1, 2]. Indications that the holographic picture could be a fundamental feature of the gravitational interaction come also from string theory and cosmology (For a recent review see [3]).

In particular, string theory allows us in a number of cases to identify and count the quantum states of the black hole and to reproduce exactly the Bekenstein-Hawking result. Moreover, explicit realizations of the holographic principle has been found for anti-de Sitter (AdS) (and de Sitter) gravity, the so-called anti-de Sitter/conformal field theory (AdS/CFT) correspondence [4, 6, 5].

Despite of this considerable progress, the deep meaning of the holographic principle remains somehow mysterious. Although everyone agrees that holography must be an essential feature of any theory of quantum gravity, its relationship with the basic principles of quantum mechanics and general relativity is poorly understood [7, 8]. Explicit realizations of the holographic principle are known only in few cases. Moreover, they always take the form of a duality, in which the form of the mapping between bulk and boundary degrees of freedom is not explicitly known.

In this paper we discuss these issues in the context of two-dimensional (2D) AdS gravity. In this case the dual boundary theory is De Alfaro-Fubini-Furlan (DFF) [9] conformal mechanics coupled with an external source and the form of the mapping between bulk and boundary degrees of freedom is explicitly known [10, 11, 12]. An other interesting feature of the model is the fact that 2D anti-de Sitter spacetime (AdS_2) allows for three different parametrizations of the spacetime [13]. One of them exhibits an event horizons, which is analogous to the acceleration horizons of Rindler spacetime [14]. We can therefore use the mapping between the bulk and boundary degrees of freedom to put in correspondence the horizons (or more in general the

spacetime structure) of the gravity theory with the quantum mechanical (and thermal) description of the boundary mechanical system. We will show that choosing appropriately the degrees of freedom, the dual boundary theory becomes either an harmonic oscillator or a free particle. In the parametrization where AdS_2 has an horizon the corresponding mechanical system is a thermal harmonic oscillator with temperature given by the Hawking temperature of the horizon.

AdS_2 is a spacetime of constant negative curvature, $R = -2\lambda^2$. It can be defined as an hyperboloid embedded in 3D Minkowski space [13]. Differently from higher dimensional cases, AdS_2 admits three different parametrizations. Using Schwarzschild coordinates the spacetime metric for two of them can be written as [14]

$$ds^2 = -\left(\lambda^2 r^2 \pm a^2\right) dt^2 + \left(\lambda^2 r^2 \pm a^2\right)^{-1} dr^2. \tag{1}$$

The third parametrization is obtained setting $a = 0$ in the previous equation. In the following we will denote these different parametrization of AdS_2 respectively as AdS_+, AdS_- and AdS_0 (Notice the change of notation with respect to Ref. [14]). The AdS_+ spacetime is full, geodetically complete, 2D AdS spacetime. It has cylindrical topology with two disconnected $r = \infty$ timelike, conformal boundaries, each of them having the topology of S^1. Conversely, the AdS_0 parametrization covers only part of the AdS hyperboloid. Only one of the two $r = \infty$ boundaries is visible and it has the topology of the line. The spacetime has an inner, null boundary at $r = 0$. Finally, the AdS_- spacetime shares with AdS_0 the $r = \infty$ boundary structure but has an event horizon at $r = a/\lambda$, whereas $r = 0$ becomes now spacelike.

The three spacetimes are locally equivalent. The metrics in Eq. (1) and that with $a = 0$ can be transformed one into the other by means of a coordinate transformation [14]. In this paper we will only need the asymptotic, $r = \infty$, form of these transformations, i.e the transformation law for the time coordinates of the $r = \infty$ boundary of the AdS spacetime. Indicating with $\tau, t, \hat{\tau}$ the timelike coordinates of, respectively, AdS_+, AdS_0 and AdS_-, we have

$$\lambda t = \tan\frac{a\lambda\tau}{2}, \quad -\frac{\pi}{a\lambda} \leq \tau \leq \frac{\pi}{a\lambda}, \quad -\infty < t < \infty, \tag{2}$$

$$\lambda t = \pm\frac{1}{a} e^{a\lambda\hat{\tau}}, \quad -\infty < \hat{\tau} < \infty, \tag{3}$$

where the $\pm$ signs hold, respectively, for $t > 0$ and $t < 0$. Notice that

choosing the + sign in Eq. (3), the time coordinate $\hat{\tau}$ of the AdS_- boundary covers only the region $t > 0$ of the AdS_0 boundary.

The relationship between AdS_- and AdS_0 is analogue to that between Rindler and Minkowski spacetime. AdS_- can be considered as the thermalization of AdS_0 at temperature, given by the Hawking temperature of the horizon, $T = a\lambda/2\pi$. Correspondingly, the vacuum for quantum fields in AdS_0 will be detected by an AdS_- observer as a thermal flux of particles at temperature $T = a\lambda/2\pi$ [14] .

Introducing a scalar field Φ (the dilaton) AdS_2 can be obtained as classical solution of the 2D gravity action $S = \frac{1}{2}\int d^2x\sqrt{-g}\Phi(R + 2\lambda^2)$. The classical solution is now described by the metric (1) endowed with a linear varying dilaton $\Phi = \lambda r$ (The most general solution contains a multiplicative integration constant Φ_0, which is irrelevant for our purposes and has been set equal to 1) . The presence of the dilaton is crucial. It enable us to interpret AdS_- as a 2D black hole. In fact Φ^{-1} is proportional to the (coordinate dependent) 2D newton constant, so that $r = 0$ can be considered as a singularity. The mass, temperature and entropy of the black hole are given by

$$M = \frac{1}{2}\lambda a^2, \quad T = \frac{a\lambda}{2\pi}, \quad S = 2\pi a. \tag{4}$$

In this context the AdS_0 spacetime has to be considered as the $M = T = 0$ solution, whereas AdS_+ describes a "naked" singularity, a black hole with negative mass $M = -(1/2)\lambda a^2$.

Two-dimensional AdS gravity induces on the spacetime boundary a conformal invariant dynamics [10]. The boundary theory has the form of de Alfaro-Fubini-Furlan (DFF) conformal mechanics coupled with an external source. Moreover, the thermodynamical entropy (4) can be exactly reproduced by counting states in the boundary conformal theory [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. The equations describing the boundary dynamics can be derived considering the asymptotic symmetry group of AdS_2 and the related large r behavior of the metric and of the dilaton

$$g_{tt} \sim -\lambda^2 r^2 + \gamma_{tt}(t), \quad g_{tt} \sim \frac{1}{\lambda^2 r^2} + \frac{\gamma_{rr}(t)}{\lambda^4 r^4}, \quad \Phi \sim \lambda\rho(t) r + \frac{\gamma_\Phi(t)}{2\lambda r}. \tag{5}$$

The fields $\gamma_{tt}, \gamma_{rr}, \gamma_\Phi, \rho$ represent boundary and dilaton deformations. The field equations for the 2D metric and dilaton projected on the boundary produce the dynamical equations [10]

$$\lambda^{-2}\ddot{\rho} - \rho\gamma + \beta \;\; = \;\; 0, \tag{6}$$

$$\dot{\rho}\gamma + \dot{\beta} \;=\; 0, \tag{7}$$

where $\gamma = \gamma_{tt} - (1/2)\gamma_{rr}$, $\beta = (1/2)\rho\gamma_{rr} + \gamma_\Phi$ and the dot denotes derivation with respect to time.

Two-dimensional dilaton gravity has no physical propagating, bulk degrees of freedom. However, Eqs. (6), (7) tell us that on the boundary of the AdS spacetime there are dynamical degrees of freedom. This phenomenon has been already observed in an other topological theory, namely 3D gravity: pure gauge bulk degrees of freedom become physical on the boundary [27]. In the case under consideration the only dynamical boundary degree of freedom is the dilaton deformation ρ. The other deformations γ, β appearing in Eqs. (6), (7) are not dynamical and are related to the diffeomorphism invariance of the 2D bulk theory. Under the action of infinitesimal diffeomorphisms of the bulk that leave the form (5) of the metric invariant, ρ, γ, β transform as conformal fields of weights $-1, 2, 1$ respectively [10].

These bulk transformations are realized on the AdS boundary as the $diff_1$ group of time reparametrizations, whose generators satisfy a Virasoro algebra [15, 16]. In principle one could then set γ and/or β to a constant just by using the diffeomorphism invariance of the bulk gravity theory. This is not possible if we want to preserve full conformal invariance of the boundary theory. Conformal invariance is crucial if one wants to use the boundary theory to give a microscopical interpretation of the thermodynamical entropy of the 2D black hole [15, 16]. In this case γ and β have to be considered as external sources that encode the information about the diffeomorphism invariance of the 2D gravity theory. This leads to the interpretation of the dynamical system (6), (7) as DFF conformal mechanics coupled to an external source [10].

In this paper we will not require invariance of the boundary theory under the $diff_1$ conformal group, so that we can hold γ constant. For $\gamma = const.$ Eq. (7) can be easily integrated to give $\beta = -\rho\gamma + C$ where C is an integration constant. Setting $C = 0$ and using this equation into Eq. (6) and one gets

$$\ddot{\rho} - 2\lambda^2 \rho\gamma = 0, . \tag{8}$$

This equation describes an elementary mechanical system. It is the equation of motion coming from the lagrangian

$$\mathcal{L} = \frac{\dot{q}^2}{2} - \frac{\omega^2}{2} q^2, \tag{9}$$

with

$$q = \frac{\rho}{\sqrt{\lambda}}, \quad \omega^2 = -2\lambda^2\gamma. \tag{10}$$

Depending on the sign of γ, the Lagrangian (9) describes a harmonic oscillator ($\gamma < 0$) a free particle ($\gamma = 0$) and a harmonic oscillator with imaginary frequency ($\gamma > 0$).

We can now easily identify the mechanical system associated with the three AdS spacetimes discussed in the previous section. AdS_+ has $\gamma = -a^2/2 = -M/\lambda$, its counterpart on the boundary is a harmonic oscillator with frequency

$$\omega = a\lambda = \sqrt{2M\lambda}. \tag{11}$$

AdS_0 is characterized by $\gamma = 0$, it corresponds to a free particle. AdS_-, the black hole, has $\gamma = a^2/2 = M/\lambda$, corresponding to a harmonic oscillator with imaginary frequency $-i\omega$, with ω given by Eq. (11).

In establishing the correspondence between the 2D metric (1) and the mechanical system (9), we have completely forgotten the global features of the boundary of the AdS spacetime. The time coordinate of the mechanical system takes its value on the timelike boundary of AdS_2. The time coordinates of the systems with $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$ are given by τ, t, and $\hat{\tau}$ respectively. They are related one with the other by the transformations (2), (3). For this reason we need a general formalism, describing the time evolution of a mechanical system, which allows for time reparametrizations such as those given in Eq. (2), (3). A general formalism with these features has been proposed by DFF in their investigations of conformal mechanics [9].

The action

$$A = \frac{1}{2}\int \dot{Q}^2 dt, \tag{12}$$

describes a free particle and is invariant under the "little" conformal group generated by translations H, dilatations D and special conformal transformations K. The most general conformal invariant model is given by the DFF Lagrangian $\mathcal{L} = (1/2)(\dot{Q}^2 - g/Q^2)$. However, in this paper we only need to consider the particular case $g = 0$. The generators obey the algebra $[H, D] = iH$, $[K, D] = -iK$, $[H, K] = 2iD$. It has been noticed [9] that any linear combination of the three generators

$$G = uH + vD + wK, \tag{13}$$

is a constant of motion, therefore can be used to generate the dynamics of the system. The generators G have been classified by DFF in three classes

depending on the sign of the determinant $\Delta = v^2 - 4uw$. For $\Delta < 0$ G is a compact operator, its spectrum is discrete and bounded from below, whereas its eigenstates are normalizable. $\Delta > 0$ corresponds to a non-compact G, whose spectrum is unbounded from below. Finally, $\Delta = 0$ corresponds to "parabolic" generators, whose spectrum is continuos and bounded from below.

DFF argued that only operators with $\Delta < 0$ lead to time evolution laws that are physically acceptable. Moreover, they can be used to solve the well-known infrared problem, which appears when time evolution is generated by parabolic generators (like H). We will show later on this paper that in our framework we can also give a physical meaning to generators with $\Delta > 0$.

The operator G generates time evolution of the system but in terms of a new time variable τ (and a new field $q(\tau)$) given by

$$d\tau = \frac{dt}{u + vt + wt^2}, \quad q(\tau) = \frac{Q(t)}{(u + vt + wt^2)^{1/2}}. \tag{14}$$

The action (12), expressed in terms of the new variables takes the form [9]

$$A = \frac{1}{2} \int d\tau \left(\dot{q}^2 + \frac{\Delta}{4} q^2 \right). \tag{15}$$

Choosing appropriately the parameters u, v, w in Eq. (13), we can obtain the lagrangian (9) from Eq. (15).

For $v = w = 0$ and $u = 1$, time evolution (with respect to the time coordinate $-\infty < t < \infty$) is generated by the operator $G = H$. $\Delta = 0$ and Eq. (15) describes the free particle of Eq. (9) (with $\omega = 0$) associated with the AdS_0 spacetime.

For $u = a, v = 0$ and $w = a\lambda^2$ the generator G is compact ($\Delta = -4a^2\lambda^2$) and given by

$$G = \lambda a(\lambda K + \lambda^{-1} H). \tag{16}$$

The operator G generates time evolution with respect to the time coordinate τ of Eq. (2). The action (15) describes the harmonic oscillator of Eq. (9), with ω given by Eq. (11), i.e the mechanical system associated with the AdS_+ spacetime. The transformation between the time coordinates t and τ can be obtained integrating the first Eq. (14). One finds that after the rescaling $\tau \to \tau/2$ this transformation matches exactly the transformation (2) (The rescaling is necessary owing to the different conformal weights of the fields q in Eq. (9) and (12), which are respectively -1 and $-1/2$).

Quantizing the classical system we get the usual quantum harmonic oscillator. In our context the spectral properties of the quantum harmonic oscillator are a simple consequence of the compactness of the time-evolution operator G and of the periodicity of the time coordinate τ

For $u = w = 0$ and $v = 2a\lambda$, $\Delta = 4a^2\lambda^2$ is positive and the operator

$$G = 2a\lambda D \tag{17}$$

is noncompact and generates time evolution with respect to the time $\hat{\tau}$ of Eq. (3). The action (15) describes the harmonic oscillator with imaginary frequency $\omega = -ia\lambda$ of Eq. (9), associated with AdS$_-$. A DFF model with a harmonic potential having the wrong sign has been also found considering the motion of a charged particle near the horizon of an extreme Reissner-Nordström solution [28]. Some related aspects of the conformal symmetry for particles moving in a AdS background have been also discussed in Ref. [29]

Integrating Eq. (14) and rescaling the time coordinate $\tau \to \hat{\tau}/2$ one gets the transformation (3) relating the time coordinates on the boundary of AdS$_-$ and AdS$_0$. The coordinate transformation (3) exchanges the generator of time translations H with that of dilatations D. For this reason it can be considered as the one-dimensional analogue of the plane-cylinder transformation for 2D conformal field theories.

Differently from AdS$_+$, the AdS$_-$ spacetime presents an event horizon. The presence of a spacetime region that is causally disconnected form the outside has a strong impact on the features of the associated mechanical system defined on the spacetime boundary. A quantum field theory in the AdS$_-$ background will generally have problems, owing to the necessity of tracing out the degrees of freedom behind the horizon. This is a well-known effect that leads to the interpretation of AdS$_-$ as the thermalization of AdS$_0$ [14]. We therefore expect that also the boundary mechanical system will be plagued by the same problem. It shows up in two related issues. First, the Lagrangian (9) in this case has a potential $V(q) = -\frac{1}{2}\omega^2 q^2$, which is unbounded from below. Correspondingly, the operator G of Eq. (17), which generates the time evolution with respect to the time coordinate $\hat{\tau}$, is non-compact. The system does not seem to have a physically reasonable spectrum. Second, an observer using the time coordinate $\hat{\tau}$ cannot see the whole history of the system as seen by an observer using the time coordinate t (or τ) because, owing to Eq (3), for $-\infty < \hat{\tau} < \infty$, $0 < t < \infty$. The operator G cannot

generate unitary time evolution over the full range of the time variable t. These troubles led DFF to discard the class of generators with $\Delta > 0$, as physically unacceptable.

In our context the above mentioned problems have a natural interpretation: they are the boundary counterpart of the presence of an horizon on the 2D bulk. We can therefore hope to give to the mechanical system (9) with $\omega^2 < 0$ a reasonable physical meaning. We will show using two different methods that the system can be interpreted as a *thermal quantum harmonic oscillator at the horizon temperature* (4).

Formally, the mechanical system under consideration is the analytic continuation $\omega \to -i\omega$ of the usual harmonic oscillator. Let us now consider the time evolution of an energy eigenstate $|u_n\rangle$, of the harmonic oscillator with eigenvalue E_n,

$$|u_n(\tau)\rangle = e^{-iE_n\tau}|u_n(0)\rangle. \tag{18}$$

Because for the harmonic oscillator E_n depends linearly on ω by analytical continuation, $\omega \to -i\omega$, we get $|u_n(\tau)\rangle = e^{-E_n\tau}|u_n(0)\rangle$. After a period $2\pi/\omega$ we get the (unnormalized) probability distribution

$$\rho_n = \langle u_n(0)|u_n(\frac{2\pi}{\omega})\rangle = e^{-\beta E_n}, \tag{19}$$

where $\beta = 2\pi/\omega = 2\pi/a\lambda$. Eq. (19) describes a thermal distribution of harmonic oscillators at temperature $T = a\lambda/2\pi$, the temperature of the horizon.

The same result can be obtained using a method which is analog to that used in quantizing free fields in the AdS_- and AdS_0 background [14]. The key point is that the time coordinate $\hat{\tau}$ covers only the region $0 \leq t < \infty$. The time evolution of a generic quantum state with respect to the time $\hat{\tau}$ is given by

$$|\psi_I(\hat{\tau})\rangle = \sum_n c_n e^{-iE_n(\hat{\tau}-\hat{\tau}_0)}|u_n(\hat{\tau}_0)\rangle, \quad a\lambda t = \exp a\lambda\hat{\tau}. \tag{20}$$

This time evolution law holds only in the region $I : t, t_0 \geq 0$, where t, t_0 are the images of $\hat{\tau}, \hat{\tau}_0$ trough the transformation (3) taken with the plus sign. We can continue Eq. (20) into the region $II : t, t_0 \leq 0$ using Eq. (3) with the minus sign and writing,

$$|\psi_{II}(\hat{\tau})\rangle = \sum_n c_n e^{-iE_n(\hat{\tau}-\hat{\tau}_0)}|u_n(\hat{\tau}_0)\rangle, \quad a\lambda t = -\exp a\lambda\hat{\tau}. \tag{21}$$

The problem is that owing to the change of sign at $t = 0$, $e^{-iE_n(\hat{\tau}-\hat{\tau}_0)}$ is not analytic at that point. As a consequence time evolution will be unitary separately on the regions I and II but will become non-unitary when we cross $t = 0$. We can recover analyticity at $t = 0$ by defining

$$|\psi(\hat{\tau})\rangle = A \sum_n c_n e^{-\frac{\pi E_n}{a\lambda}} e^{-iE_n(\hat{\tau}-\hat{\tau}_0)} |u_n(\hat{\tau}_0)\rangle, \tag{22}$$

where A is a normalization constant and now $\hat{\tau}$ and $\hat{\tau}_0$ belong respectively to the regions I and II. Time evolution is now not unitary but analytic at $t = 0$ because the time evolution factor in Eq. (22) is proportional to $(t)^{-iE_n/a\lambda}$. If the system is in an eigenstate $|u_n\rangle$ at the time $\hat{\tau}_0$, the probability of finding it in the same state at the time $\hat{\tau}$ will be given by

$$P_n = |A|^2 e^{-\frac{2\pi E_n}{a\lambda}}. \tag{23}$$

The normalization constant can be given in terms of the partition function Z, so that we finally find

$$P_n = \frac{e^{-\beta E_n}}{Z}, \tag{24}$$

i.e a thermal distribution at temperature given by the Hawking temperature of the horizon, $T = a\lambda/2\pi$.

Our simple example tells us that the holographic principle, quantum mechanics and the coarse graining of information of the thermal description are intimately intertwined with the presence of an horizon and with the topology of the spacetime. On the one hand our results give support to the usual thermal interpretation of horizons. Constructing an holographic dual of the horizon in terms of an elementary mechanical system, we have found it still has the features of a thermal ensemble: a thermal ensemble of harmonic oscillators.

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